Int. J. Solids Structures Vol. 18, No. 5, pp. 411–418, 1982
Printed in Great Britain.
0020-7683/82/050411–08$03.00/0
Pergamon Press Ltd.

BIFURCATION IN AN ELASTIC PLATE ON A RIGID SUBSTRATE

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(*Received 7 May* 1981; *in revised form* 30 *July* 1981)

Abstract—An infinite plate of neo-Hookean elastic material is bonded on one face to a rigid substrate. It is subjected to a uniform shear and dead-loaded with a uniform thrust. A periodic bifurcation solution is obtained when the thrust per unit area exceeds a critical value. The relation between the wave-length, thrust, amount of shear and plate thickness is obtained.

1. INTRODUCTION

In this paper we consider the critical loading conditions for which a bifurcation solution is obtained when an infinite plate of incompressible isotropic neo-Hookean elastic material, bonded on one face to a rigid substrate, is subjected to a uniform shear of amount K and simultaneously dead-loaded on the other face by a uniform normal thrust.

We suppose that an infinitesimal spatially periodic deformation is superposed. A secular equation is obtained for the determination of the wave-length of the superposed deformation. This secular eauation yields a real value for the wave-length when some critical value of the normal thrust per unit area, which depends slightly on K, and is approximately equal to twice the shear modulus, is reached. For this critical value of the thrust, the wave-length of the infinitesimal superposed deformation is zero and, as the thrust is increased beyond this value, K remaining fixed, the wave-length increases.

Except at values of the thrust per unit area near the critical value, the wave-length is proportional to the square root of the thrust per unit area and is nearly independent of K. At values of the thrust near the critical value, the wavelength becomes extremely sensitive to the thrust. At all values of the thrust and of K, the wave-length is proportional to the thickness of the plate, as may be expected from dimensional considerations.

For specified values of the thickness and of the thrust per unit area, beyond the critical value of the latter, a periodic static bifurcation solution is obtained with uniquely determined wave-length. However, if the plate were finite in the direction of the periodicity, the end conditions would enable us to determine the spectrum of values of the wave-length, and hence of the thrust per unit area, at which the bifurcation solution can occur.

2. BASIC EQUATIONS

We consider a flat plate of incompressible isotropic neoHookean elastic material of thickness h to be located with its major surfaces normal to the 2-axis of a rectangular Cartesian coordinate system x. The dimensions of the plate in the 1 and 3 directions are supposed large compared with h.

Let $\boldsymbol{\xi}$ be the vector position of a generic particle of the plate in its undeformed state and let ξ_α ($\alpha = 1, 2, 3$) be the components of $\boldsymbol{\xi}$ in the system x. We suppose that initially the plate is located with its major surfaces in the planes $\xi_2 = 0$ and $\xi_2 = h$ and that the face $\xi_2 = 0$ is bonded to a rigid plate which remains fixed in space.

Suppose the elastic plate undergoes a deformation in which a particle initially at $\boldsymbol{\xi}$ moves to vector position $\mathbf{x}$ with components x_i ($i = 1, 2, 3$) in the system x. Then, the strain-energy W per unit volume is given, in appropriately chosen units, by

$$W = \frac{1}{2}(x_{i,\alpha}x_{i,\alpha} - 3), \tag{2.1}$$

where $,\alpha$ is used to denote differentiation with respect to ξ_α. The Piola–Kirchhoff stress $\bar{\Pi}_{\alpha i}$, referred to the system x, is given by

$$\bar{\Pi}_{\alpha i} = x_{i,\alpha} - \frac{1}{2}\bar{P}\epsilon_{ijk}\epsilon_{\alpha\beta\gamma}x_{j,\beta}x_{k,\gamma}, \tag{2.2}$$

where ϵ_{ijk} denotes the alternating symbol and $\bar{P}$ is an arbitrary hydrostatic pressure. Since the material considered is incompressible

$$\det|x_{i,\alpha}| = 1. \tag{2.3}$$

We now suppose that the deformation $\boldsymbol{\xi} \to \mathbf{x}$ is the resultant of a finite deformation $\boldsymbol{\xi} \to \mathbf{X}$ and an infinitesimal deformation $\mathbf{X} \to \mathbf{x}$, where

$$\mathbf{x} = \mathbf{X} + \epsilon\mathbf{u} \tag{2.4}$$

and ϵ is a small parameter. We assume that the force system associated with the deformation $\boldsymbol{\xi} \to \mathbf{x}$ differs by terms of order ϵ from that associated with the deformation $\boldsymbol{\xi} \to \mathbf{X}$. We accordingly write

$$\bar{\Pi}_{\alpha i} = \bar{\Pi}_{\alpha i} + \epsilon\pi_{\alpha i}, \quad \bar{P} = p + \epsilon p. \tag{2.5}$$

Then, from (2.2) we obtain, with (2.4) and (2.5),

$$\Pi_{\alpha i} = X_{i,\alpha} - \frac{1}{2}P\epsilon_{ijk}\epsilon_{\alpha\beta\gamma}X_{j,\beta}X_{k,\gamma},$$

$$\pi_{\alpha i} = u_{i,\alpha} - \frac{1}{2}\epsilon_{ijk}\epsilon_{\alpha\beta\gamma}\{P(X_{k,\gamma}u_{j,\beta} + X_{j,\beta}u_{k,\gamma}) + pX_{j,\beta}X_{k,\gamma}\}. \tag{2.6}$$

Similarly from (2.3) we have

$$\det|X_{i,\alpha}| = 1, \quad \epsilon_{ijk}\epsilon_{\alpha\beta\gamma}X_{i,\alpha}X_{j,\beta}u_{k,\gamma} = 0. \tag{2.7}$$

The Piola–Kirchhoff equations of equilibrium yield

$$\Pi_{\alpha i,\alpha} = 0 \text{ and } \pi_{\alpha i,\alpha} = 0. \tag{2.8}$$

3. THE GOVERNING EQUATIONS

If the deformation $\boldsymbol{\xi} \to \mathbf{X}$ is a simple shear of amount K, for which the direction of shear is the 1-direction and the plane of shear is the 12-plane, then

$$X_1 = \xi_1 + K\xi_2, \quad X_2 = \xi_2, \quad X_3 = \xi_3. \tag{3.1}$$

We shall assume that the superposed infinitesimal deformation $\mathbf{X} \to \mathbf{x}$ is a plane deformation in the 12-plane. We can then write

$$u_1 = u_1(\xi_1, \xi_2), \quad u_2 = u_2(\xi_1, \xi_2), \quad u_3 = 0. \tag{3.2}$$

With (3.1), eqn $(2.7)_1$ is automatically satisfied and with (3.1) and (3.2) eqn $(2.7)_2$ yields

$$u_{1,1} + u_{2,2} - Ku_{2,1} = 0. \tag{3.3}$$

Also, with (3.1), (3.2) and (3.3) eqns (2.6) yield

$$\Pi_{11} = \Pi_{22} = \Pi_{33} = 1 - P,\ \Pi_{12} = KP,\ \Pi_{21} = K,$$
$$\Pi_{31} = \Pi_{13} = \Pi_{23} = \Pi_{32} = 0, \tag{3.4}$$

and

$$\pi_{11} = u_{1,1} - Pu_{2,2} - p,\ \pi_{22} = u_{2,2} - Pu_{1,1} - p,\ \pi_{3,3} = -p,$$
$$\pi_{12} = u_{2,1} + Pu_{1,2} + pK,\ \pi_{21} = u_{1,2} + Pu_{2,1},\ \pi_{31} = \pi_{13} = \pi_{23} = \pi_{32} = 0. \tag{3.5}$$

With (3.4), eqn $(2.8)_1$ implies that P is constant. With (3.5), the incremental equation of equilibrium $(2.8)_2$ yields

$$u_{1,11} + u_{1,22} = p_{,1},\ u_{2,11} + u_{2,22} + Kp_{,1} = p_{,2},\ p_{,3} = 0. \tag{3.6}$$

The last of these equations implies that $p = p(\xi_1, \xi_2)$.

From (3.3) it follows that there exists a function $\psi(\xi_1, \xi_2)$ in terms of which we may express u_1, u_2 by the relations

$$u_1 = \psi_{,2} - K\psi_{,1},\ u_2 = -\psi_{,1}. \tag{3.7}$$

By substituting (3.7) in $(3.6)_{1,2}$ and eliminating p, we obtain

$$\nabla^2[(1 + K^2)\psi_{,11} + \psi_{,22} - 2K\psi_{,12}] = 0, \tag{3.8}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial \xi_1{}^2} + \frac{\partial^2}{\partial \xi_2{}^2}. \tag{3.9}$$

We shall obtain solutions of (3.8) which are sinusoidal in the 1-direction. Accordingly, with the usual complex notation, we write

$$\psi = \phi(\xi_2)\, e^{ik\xi_1}, \tag{3.10}$$

where k is a constant. Introducing (3.10) into (3.8), we obtain

$$\phi'''' - 2ikK\phi''' - k^2(2 + K^2)\phi'' + 2ik^3K\phi' + k^4(1 + K^2)\phi = 0, \tag{3.11}$$

where the prime denotes differentiation with respect to ξ_2. Equation (3.11) has the general solution

$$\phi = \sum_{A=1}^{4} a_A\, e^{\alpha_A \xi_2}, \tag{3.12}$$

where the a's are (complex) constants and the α's are given by

$$\alpha_1 = k,\ \alpha_2 = -k,\ \alpha_3 = k(1 + iK),\ \alpha_4 = k(-1 + iK). \tag{3.13}$$

With (3.10) and (3.12), we obtain from (3.7)

$$u_1 = e^{ik\xi_1} \sum_{A=1}^{4} (\alpha_A - ikK) a_A\, e^{\alpha_A \xi_2},$$
$$u_2 = -ik\, e^{ik\xi_1} \sum_{A=1}^{4} a_A\, e^{\alpha_A \xi_2}. \tag{3.14}$$

With (3.14), eqns (3.6) yield

$$p_{,1} = e^{ik\xi_1} \sum_{A=1}^{4} (\alpha_A{}^2 - k^2)(\alpha_A - ikK)a_A\, e^{\alpha_A \xi_2},$$

$$p_{,2} = e^{ik\xi_1} \sum_{A=1}^{4} (\alpha_A{}^2 - k^2)[-ik(1+K^2) + K\alpha_A]a_A\, e^{\alpha_A \xi_2}. \tag{3.15}$$

These equations yield

$$p = -\frac{i}{k} e^{ik\xi_1} \sum_{A=1}^{4} (\alpha_A{}^2 - k^2)(\alpha_A - ikK)a_A\, e^{\alpha_A \xi_2} + \text{constant}. \tag{3.16}$$

With (3.14) and (3.16), we obtain from (3.5)

$$\pi_{11} = e^{ik\xi_1} \sum_{A=1}^{4} \left\{ik\alpha_A\left(P + \frac{\alpha_A{}^2}{k^2}\right) + K\alpha_A{}^2\right\} a_A\, e^{\alpha_A \xi_2},$$

$$\pi_{22} = e^{ik\xi_1} \sum_{A=1}^{4} \left\{-ik\alpha_A\left(2 + P - \frac{\alpha_A{}^2}{k^2}\right) + K(\alpha_A{}^2 - k^2 - k^2 P)\right\} a_A\, e^{\alpha_A \xi_2},$$

$$\pi_{12} = e^{ik\xi_1} \sum_{A=1}^{4} \left\{-ikK\alpha_A\left(P - 1 + \frac{\alpha_A{}^2}{k^2}\right) + k^2(1+K^2) + \alpha_A{}^2(P - K^2)\right\} a_A\, e^{\alpha_A \xi_2}, \tag{3.17}$$

$$\pi_{21} = e^{ik\xi_1} \sum_{A=1}^{4} \{-ikK\alpha_A + (\alpha_A{}^2 + k^2 P)\} a_A\, e^{\alpha_A \xi_2}.$$

4. SOLUTION OF THE SECULAR EQUATION

In this section we shall assume that the surface conditions on the surface $\xi_2 = h$ are dead-loading conditions. Accordingly,

$$\pi_{22}|_{\xi_2 = h} = \pi_{21}|_{\xi_2 = h} = 0. \tag{4.1}$$

On the surface $\xi_2 = 0$, we have

$$u_1|_{\xi_2 = 0} = u_2|_{\xi_2 = 0} = 0. \tag{4.2}$$

Introducing (3.14) into (4.2), we obtain

$$\sum_{A=1}^{4} (\alpha_A - ikK)a_A = 0, \quad \sum_{A=1}^{4} a_A = 0. \tag{4.3}$$

Again introducing $(3.17)_{2,4}$ into (4.1), we obtain

$$\sum_{A=1}^{4} B_A a_A = 0, \quad \sum_{A=1}^{4} C_A a_A = 0, \tag{4.4}$$

where

$$B_A = \left\{-ik\alpha_A\left(2 + P - \frac{\alpha_A{}^2}{k^2}\right) + K(\alpha_A{}^2 - k^2 - k^2 P)\right\} e^{\alpha_A h},$$

$$C_A = \left\{-ikK\alpha_A + \alpha_A{}^2 + k^2 P\right\} e^{\alpha_A h}. \tag{4.5}$$

The necessary and sufficient condition for (4.3) and (4.4) to have a non-trivial solution for a_A is

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ B_1 & B_2 & B_3 & B_4 \\ C_1 & C_2 & C_3 & C_4 \end{vmatrix} = 0. \tag{4.6}$$

With the expressions (4.5) for B_A and C_A, the secular equation (4.6) can be rewritten (see Appendix) as

$$\beta_1 Q^2 + 2\beta_2 Q + \beta_3 = 0, \tag{4.7}$$

where

$$Q = 1 + P,$$

$$\begin{aligned} \beta_1 &= (K^2+4) - (K^2 \cosh 2\mu + 4\cos K\mu), \\ \beta_2 &= 2K^2(\cosh 2\mu - \cos K\mu), \\ \beta_3 &= K^2[(K^2+4) + (K^2\cosh 2\mu + 4\cos K\mu)], \end{aligned} \tag{4.8}$$

with

$$\mu = kh. \tag{4.9}$$

From (4.7), Q is given by

$$Q = -\frac{1}{\beta_1}\{\beta_2 \pm (\beta_2{}^2 - \beta_1\beta_3)^{1/2}\}. \tag{4.10}$$

We note from (4.8) that

$$\begin{aligned} \beta_2{}^2 - \beta_1\beta_3 &= K^2(K^2+4)(K^2\sinh^2 2\mu - 4\sin^2 K\mu), \\ \beta_1 &= -2\left(K^2\sinh^2\mu - 4\sin^2\frac{1}{2}K\mu\right), \\ \beta_3 &= 2K^2\left(K^2\cosh^2\mu + 4\cos^2\frac{1}{2}K\mu\right). \end{aligned} \tag{4.11}$$

Since $K^2\sinh^2 2\mu \geq 4\sin^2 K\mu$ for all K and μ, it follows from (4.10) and (4.11), that Q is real for all K and μ. Also, it is evident from $(4.11)_{2,3}$ that $\beta_1 \leq 0$ and $\beta_3 \geq 0$. It follows that the negative alternative in (4.10) leads to a negative value for Q. Since, from (3.4), the condition for the normal traction on $\xi_2 = h$ to be a thrust is $P > 1$, i.e. $Q > 2$, it follows that the negative alternative in (4.10) corresponds to the normal traction on $\xi_2 = h$ being tensile.

We note also that $\beta_2 \geq 0$. Accordingly, the necessary and sufficient condition that $Q > 2$, i.e. the normal traction is a thrust, is

$$(\beta_2{}^2 - \beta_1\beta_3)^{1/2} > -2\beta_1 - \beta_2. \tag{4.12}$$

From (4.8) it follows that

$$2\beta_1 + \beta_2 = 2(K^2+4)(1-\cos K\mu). \tag{4.13}$$

Accordingly, apart from the trivial case when $\cos K\mu = 1$, the inequality (4.12) is always satisfied by the positive alternative in (4.10). We conclude that this corresponds to the normal force on the surface $\xi_2 = h$ being a thrust. In the remainder of this paper we consider only this situation.

In Fig. 1 we plot from (4.10), with (4.8) and (4.11), the values of the thrust $N = P - 1 = Q - 2$ against μ for K fixed. The curve in Fig. 1 was drawn for $K = 0.4$. However, the dependence of N on K, for K in the range 0 to 0.4, is less than 2%. It was found from the numbers from which the figure was plotted that $Q - 2 \approx 13.2/\mu^2$ for $\mu < 1$.

It is instructive to consider three limiting cases.

(a) *μ fixed*, K→0

From (4.8) it follows that

$$\begin{aligned} \beta_1 &= 2K^2(\mu^2 - \sinh^2\mu) + 0(K^4), \\ \beta_2 &= 4K^2\sinh^2\mu + 0(K^4), \\ \beta_3 &= 8K^2 + 0(K^4). \end{aligned} \tag{4.14}$$

With these expressions we obtain from (4.10)

$$Q = \frac{2\{\sinh^2\mu + (\sinh^4\mu + \sinh^2\mu - \mu^2)^{1/2}\}}{\sinh^2\mu - \mu^2}. \tag{4.15}$$

In the limit $\mu \to \infty$, $Q = 4$.
In the limit $\mu \to 0$, $Q = (6/\mu^2)(1 + (2/\sqrt{3}))$.

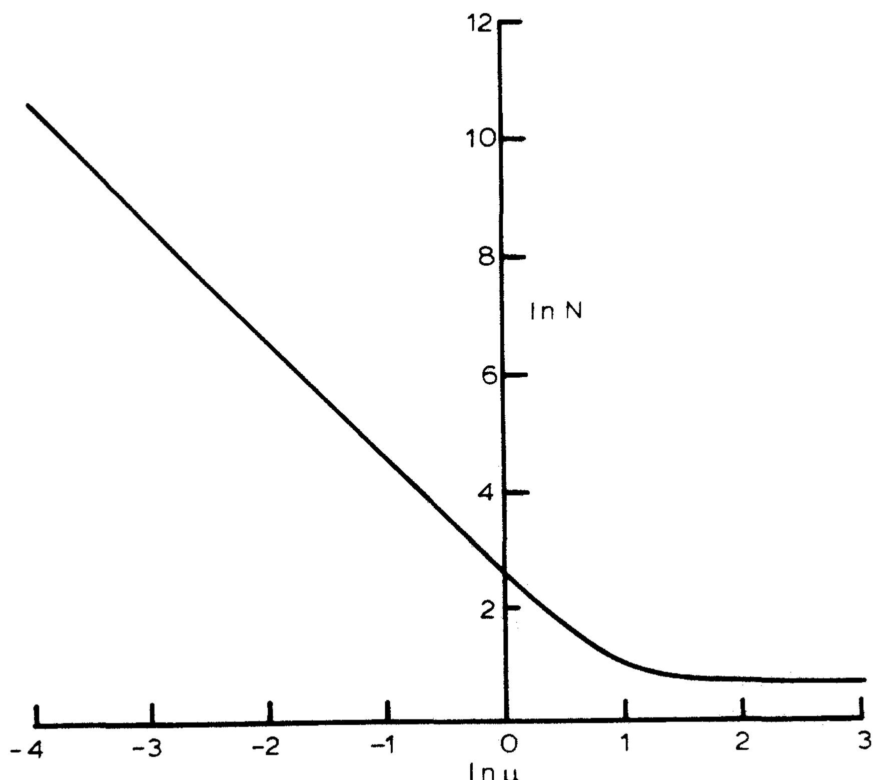

Fig. 1. ln N vs ln μ for $K = 0.4$.

(b) K *fixed*, $\mu \to \infty$

From (4.8), it follows that, as $\mu \to \infty$,

$$\beta_1 = -\frac{1}{2}K^2 e^{2\mu},\quad \beta_2 = K^2 e^{2\mu},\quad \beta_3 = \frac{1}{2}K^4 e^{2\mu}. \tag{4.16}$$

Then, from (4.10), we obtain

$$Q = 2 + (K^2 + 4)^{1/2}. \tag{4.17}$$

As K increases from 0 to 1, $Q|_{\mu=\infty}$ increases from 4 to 4.236.

(c) K *fixed*, $\mu \to 0$

From (4.8), we obtain, as $\mu \to 0$

$$\beta_1 = -\frac{1}{6}\mu^4 K^2(K^2+4) + 0(\mu^6),\quad \beta_2 = \mu^2 K^2(K^2+4) + 0(\mu^4),\quad \beta_3 = 2K^2(K^2+4) + 0(\mu^2). \tag{4.18}$$

Then, from (4.10), we obtain

$$Q = \frac{6}{\mu^2}\left(1 + \frac{2}{\sqrt{3}}\right) \approx 12.93/\mu^2, \tag{4.19}$$

in agreement with the result obtained in case (a).

It is seen from the asymptotic results in (a) that a static bifurcation solution becomes possible only when the thrust Q per unit area reaches some critical value. This value depends only very slightly on the amount of imposed shear and increases from 4 to 4.236 as the amount of shear increases from 0 to 1. At the critical value of the thrust, $\mu = kh = \infty$, i.e. the wave-length of the bifurcation solution is zero.

It is seen from Fig. 1 that corresponding to any value of Q greater than the critical value a uniquely determined static bifurcation solution exists. The wave-length for this solution increases very rapidly as the thrust is increased beyond the critical value until the corresponding wave-length is approximately 2.31 times the thickness of the plate. For still higher values of the thrust the wave-length of the bifurcation solution increases very nearly proportionately to the square root of the thrust. For any given value of the thrust the wave-length of the static bifurcation is proportional to the thickness of the plate.

Acknowledgement—This paper was written with the partial support of the Office of Naval Research under contract 00014-76-C-0235 with Lehigh University.

APPENDIX

With the expressions (3.13) for α_A, eqn (4.6) can be written as

$$(B_1C_2 - B_2C_1) + (B_2C_3 - B_3C_2) + (B_3C_4 - B_4C_3) + (B_4C_1 - B_1C_4) + \frac{1}{2}\iota K\{(B_1 - B_2)(C_3 - C_4) - (C_1 - C_2)(B_3 - B_4)\} = 0. \tag{5.1}$$

Also, with (3.13) and the notation $Q = 1 + P$ and $\mu = kh$, we obtain from (4.5)

$$B_1 = k^2[K(1-Q) - \iota Q]\,e^{\mu},$$
$$B_2 = k^2[K(1-Q) + \iota Q]\,e^{-\mu},$$
$$B_3 = -k^2[\iota Q + K(1+\iota K)]\,e^{(1+\iota K)\mu},$$

$$
\begin{aligned}
B_4 &= k^2\{\iota Q - K(1-\iota K)\}\,\mathrm{e}^{-(1-\iota K)\mu}, \\
C_1 &= k^2(Q-\iota K)\,\mathrm{e}^{\mu}, \\
C_2 &= k^2(Q+\iota K)\,\mathrm{e}^{-\mu}, \\
C_3 &= k^2(Q+\iota K)\,\mathrm{e}^{(1+\iota K)\mu}, \\
C_4 &= k^2(Q-\iota K)\,\mathrm{e}^{-(1-\iota K)\mu}.
\end{aligned} \tag{5.2}
$$

From (5.2) we obtain

$$
\begin{aligned}
B_3C_4 - B_4C_3 &= (B_1C_2 - B_2C_1)\,\mathrm{e}^{2\iota k\mu} = -2\iota k^4\{Q + (Q-1)K^2\}\,\mathrm{e}^{2\iota k\mu}, \\
B_2C_3 - B_3C_2 &= k^4(Q+\iota K)\{2\iota Q + K(1-Q) + K(1+\iota K)\}\,\mathrm{e}^{\iota K\mu}, \\
B_4C_1 - B_1C_4 &= k^4(Q-\iota K)\{2\iota Q - K(1-Q) - K(1-\iota K)\}\,\mathrm{e}^{\iota K\mu}.
\end{aligned} \tag{5.3}
$$

Whence,

$$
(B_2C_3 - B_3C_2) + (B_4C_1 - B_1C_4) = 4\iota k^4(Q^2+K^2)\,\mathrm{e}^{\iota K\mu}. \tag{5.4}
$$

We also obtain from (5.2)

$$
(B_1-B_2)(C_3-C_4) - (C_1-C_2)(B_3-B_4) = k^4\{[QK(4-Q)+K^3](\mathrm{e}^{2\mu}+\mathrm{e}^{-2\mu}) + 2K(Q^2+K^2)\}\,\mathrm{e}^{\iota K\mu}. \tag{5.5}
$$

With (5.3), (5.4) and (5.5), the secular equation (5.1) may be rewritten as

$$
-2\{Q^2+(Q-1)K^2\}(1+\mathrm{e}^{2\iota K\mu}) + 4(Q^2+K^2)\,\mathrm{e}^{\iota K\mu} + \frac{1}{2}K^2\{[Q(4-Q)+K^2](\mathrm{e}^{2\mu}+\mathrm{e}^{-2\mu}) + 2(Q^2+K^2)\}\,\mathrm{e}^{\iota K\mu} = 0. \tag{5.6}
$$

From this the relation (4.7) with (4.8) is easily obtained.